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The matricant method application to the analysis of separatrixes splitting in the attitude dynamics of gyrostat satellites

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***Abstract*— The article considers the free motion of a single-rotor gyrostat-satellite under the action of internal harmonic perturbations. The splitting of the heteroclinic phase trajectories is considered at the action of periodic perturbations with arising the dynamical chaos. The matricant method is used to analysis of the splitting of perturbed manifolds of heteroclinic phase trajectories. A numerical method for constructing the evolution of phase trajectories based on numerical integration of differential equations of motion is implemented.**

Keywords*— *gyrostat-satellite, heteroclinic trajectory, chaotic dynamics, matricant method.

I. INTRODUCTION

Gyrostat-satellites represent a large class of spacecraft used in various fields, including orbital communications, navigation, and remote sensing of the Earth, as well as scientific experiments in space that require precise control of their orientation. The work focuses on a dual spin gyrostat-satellite, which is of interest from the perspective of nonlinear mechanics. Researching such motion modes is necessary when designing real space systems, since small inaccuracies in the system's initial parameters increase over time, making the process unpredictable. This can lead to undesirable consequences in the context of space missions, including failure to achieve objectives.

The main cause of chaotic motion modes is the multiple intersection of the split manifolds of heteroclinic trajectories [1]. Modern approaches to studying the emergence of heteroclinic trajectory splitting are based on the Melnikov-Wiggins method [2–6]. The foundations of this formalism were laid in the works of V.K. Melnikov [2] and V.I. Arnold [3], then reinterpreted by V.V. Kozlov [4] and further developed in the research of P.J. Holmes & J.E. Marsden [5], as well as S. Wiggins [6]. The above methods are well suited to homoclinic trajectories when applied correctly, but difficulties arise when analysing heteroclinic splitting. Therefore, an alternative method is required that can be correctly applied to any analysis of homoclinic and heteroclinic trajectory splitting. In this study, the matricant method to analyze heteroclinic splitting [7] is developed.

II. FORMULATION OF THE PROBLEM

This paper aims to analyse the heteroclinic splitting of phase trajectories in the dynamics of a gyrostat-satellite using the matricant method. When subjected to periodic perturbations, the trajectories split into stable and unstable manifolds that may intersect, thereby initiating heteroclinic chaos. The paper considers internal small harmonic perturbations, which can be caused by errors in angular velocity sensors and interference in the rotor motor's electrical circuits.

III. MATHEMATICAL MODEL

We consider the free motion of a system comprising a carrier body and a rotor. The carrier body has a three-axis inertia tensor and the rotor is dynamically symmetric. Let us introduce a coordinate system: The inertial coordinate system is $OXYZ$, the rotor's coordinate system is $Ox_1y_1z_1$ and the carrier body's coordinate system is $Ox_2y_2z_2$ The axes Oz_1 and Oz_2 coincide.

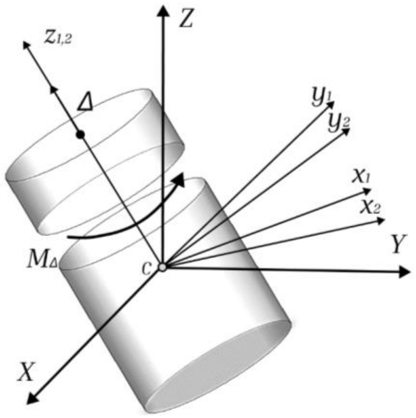

Fig. 1. System of coaxial bodies

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The system in question is described by the following Euler equations:

$$\begin{cases} I_x\dot{p} + \left(I_{2z} - I_y\right)qr + q\Delta = 0 \\ I_y\dot{q} + (I_x - I_{2z})pr - p\Delta = 0 \\ I_{2z}\dot{r} + \Delta + \left(I_y - I_x\right)pq = 0 \\ \dot{\Delta} = \mu\sin[\omega_\Delta t] \end{cases} \tag{1}$$

Where $I_2 = diag[I_{2x},\ I_{2y},\ I_{2z}]$ is the inertia tensor of the carrier body and $I_1 = diag[I_{1x},\ I_{1y},\ I_{1z}]$ is the inertia tensor of the rotor. The components of the angular velocity of the carrier body are p, q and r. $I_x = I_{1x} + I_{2x},\ I_y = I_{1x} + I_{2y},\ I_z = I_{1z} + I_{2z}$ are the principal moments of inertia of the system of coaxial bodies. M_Δ is the internal moment of forces and Δ - the longitudinal angular moment of the rotor.

The fourth equation in system (1) yields the following analytical solution:

$$\Delta(t) = \bar{\Delta} + \frac{\mu}{\omega_\Delta}\,(1 - cos[\omega_\Delta t]) \tag{2}$$

Where $\bar{\Delta}$ - the integration constant. Substituting solution (2) into system (1) yields:

$$\begin{cases} I_x\dot{p} + \left(I_{2z} - I_y\right)qr + q\bar{\Delta} = -\varepsilon I_{2z}\omega_\Delta(1 - \cos(\omega_\Delta t))q; \\ I_y\dot{q} + (I_x - I_{2z})pr - p\bar{\Delta} = \varepsilon I_{2z}\omega_\Delta\,(1 - \cos(\omega_\Delta t))p; \\ I_{2z}\dot{r} + \left(I_y - I_x\right)pq = -\varepsilon I_{2z}\omega_\Delta^2\sin(\omega_\Delta t)p\,; \end{cases} \tag{3}$$

Where ε - a small dimensionless parameter.

$$\varepsilon = \frac{\mu}{I_{2z}\omega_\Delta^2} \tag{4}$$

In the absence of perturbations, there are explicit analytical heteroclinic solutions $\{\bar{p}(t), \bar{q}(t), \bar{r}(t)\}$ corresponding to motion along heteroclinic trajectories [1].

Chaotic dynamics are studied using Poincaré sections, which depict the system's phase space as a set of discrete points corresponding to physical time values that are multiples of the perturbation period ±T. If we consider the heteroclinic trajectory separately, we obtain its split manifolds as a set of images of all points taken at one mapping step, where unstable manifolds correspond to the forward direction of time, and stable manifolds correspond to the backward direction. Let us consider the initial stages of the Poincaré section, as the fundamental forms of the split manifolds are presented here, generating all subsequent images of the manifolds.

We find the solutions of the perturbed system (3) to the neighbourhood of one of the heteroclinic trajectories in the form:

$$\dot{x} = \bar{x}(t) + \varepsilon\tilde{x}(t), \quad x = \{\bar{p}(t), \bar{q}(t), \bar{r}(t)\}^T$$

$$\begin{cases} \dot{p} = \bar{p}(t) + \varepsilon\tilde{p}(t) \\ \dot{q} = \bar{q}(t) + \varepsilon\tilde{q}(t) \\ \dot{r} = \bar{r}(t) + \varepsilon\tilde{r}(t) \end{cases} \tag{5}$$

By substituting (5) into equation (3) and linearising with respect to the small parameters, we obtain the following first-order equation for the deviations $\tilde{x}$ (t):

$$\frac{d}{dt}\begin{pmatrix}\tilde{p}\\ \tilde{q}\\ \tilde{r}\end{pmatrix} = \begin{pmatrix} 0 & A_{12}(t) & A_{13}(t) \\ A_{21}(t) & 0 & A_{23}(t) \\ A_{31}(t) & A_{32}(t) & 0 \end{pmatrix}\begin{pmatrix}\tilde{p}\\ \tilde{q}\\ \tilde{r}\end{pmatrix} + \begin{pmatrix}f_1\\ f_2\\ f_3\end{pmatrix} \tag{6}$$

where $A\left(\bar{x}(t)\right)$ and $f(\bar{x}, t)$ are functional matrices.

$$A\left(\bar{x}(t)\right) = \begin{pmatrix} 0 & -\frac{((I_{2z}-I_y)\bar{r}+\bar{\Delta})}{I_x} & -\frac{(I_{2z}-I_y)\bar{q}}{I_x} \\ -\frac{((I_x-I_{2z})\bar{r}+\bar{\Delta})}{I_y} & 0 & -\frac{(I_x-I_{2z})\bar{p}}{I_y} \\ -\frac{\bar{q}(I_y-I_x)}{I_{2z}} & -\frac{\bar{p}(I_y-I_x)}{I_{2z}} & 0 \end{pmatrix},$$

$$f(\bar{x}, t) = \begin{bmatrix} -\dfrac{\bar{q}(1 - \cos(\omega_\Delta))I_{2z}\omega_\Delta}{A} \\ \dfrac{\bar{p}(1 - \cos(\omega_\Delta))I_{2z}\omega_\Delta}{B} \\ -\sin(\omega_\Delta)\,I_{2z}\omega_\Delta^2 \end{bmatrix}. \tag{7}$$

For linear differential equations (5) with time-dependent coefficients $(A\left(\bar{x}(t)\right))$, there is a classical solution method involving a matricant [7]:

$$\begin{aligned} \tilde{x}(t) = \Omega_{t_0}^t\left(A\left(\bar{x}(t)\right)\right)\cdot \\ \cdot\int_{t_0}^t\left\{\left[\Omega_{t_0}^\tau\left(A\left(\bar{x}(t)\right)\right)\right]^{-1}\cdot f(\bar{x}(\tau), \tau)\right\}d\tau, \end{aligned} \tag{8}$$

where $\Omega_{t_0}^\tau\left(A\left(\bar{x}(t)\right)\right)$is the matricant.

Matricant can be represented as the sum of a matrix series [7]:

$$\begin{aligned} \Omega_{t_0}^t\left(A\left(\bar{x}(t)\right)\right) = E + \int_{t_0}^t A\left(\bar{x}(t)\right)dt + \\ + \int_{t_0}^t A\left(\bar{x}(t)\right)dt\int_{t_0}^t A\left(\bar{x}(t)\right)dt + \cdots \end{aligned} \tag{9}$$

The parameters $\bar{t}_{unst}$ and $\bar{t}_{st}$ are used for the explicit writing of expressions for constructing split manifolds.

$$\begin{cases} x_{unst}(\bar{t}_{unst}) = \bar{x}(\bar{t}_{unst}) + \varepsilon\tilde{x}(\bar{t}_{unst}) \\ \tilde{x}_{unst}(t_{unst}) = \Omega_0^T\left(A\left(\bar{x}(t - T + t_{unst})\right)\right)\cdot \\ \cdot\int_{t_0}^t\left\{\left[\Omega_0^\tau\left(A\left(\bar{x}(\tau - T + t_{unst})\right)\right)\right]^{-1}\cdot\right. \\ \left.\cdot f(\bar{x}(\tau - T + t_{unst}), \tau)\right\}d\tau \end{cases} \tag{10}$$

$$\begin{cases} x_{st}(\bar{t}_{st}) = \bar{x}(\bar{t}_{st}) + \varepsilon\tilde{x}(\bar{t}_{st}) \\ \tilde{x}_{st}(t_{st}) = \Omega_0^T\left(A\left(\bar{x}(t_{st} - t + T)\right)\right)\cdot \\ \cdot\int_{t_0}^t\left\{\left[\Omega_0^\tau\left(A\left(\bar{x}(t_{st} - \tau + T)\right)\right)\right]^{-1}\cdot\right. \\ \left.\cdot f(\bar{x}(t_{st} - \tau + T), \tau)\right\}d\tau \end{cases} \tag{11}$$

Expressions (10) and (11) are exact analytical solutions to the first-order system (3). Due to the difficulty of calculating the matrix analytically, a numerical procedure for determining the matrix will be implemented. The integral on the right-hand side of equations (10) and (11) can be formally expressed using the fourth-order Newton–Cotes composite rule:

$$\begin{aligned}\hat{I}(\bar{t}) = \frac{2h}{45}\sum_{j=1}^{\frac{i}{4}}\Big\{ & 7\cdot\left[\hat{\Omega}_0^{h\cdot(j+3j)}\right]^{-1}\hat{f}\big(t,h\cdot(j+3j)\big)+\\ & +32\left[\hat{\Omega}_0^{h\cdot(j+3j+1)}\right]^{-1}\hat{f}\big(t\cdot(j+3j+1)\big)+\\ & +12\left[\hat{\Omega}_0^{h\cdot(j+3j+2)}\right]^{-1}\hat{f}\big(t,h\cdot(j+3j+2)\big)+\\ & +32\left[\hat{\Omega}_0^{h\cdot(j+3j+3)}\right]^{-1}\hat{f}\big(t,h\cdot(j+3j+3)\big)+\\ & +7\left[\hat{\Omega}_0^{h\cdot(j+3j+4)}\right]^{-1}\hat{f}(t,h\cdot(j+3j+4))\Big\}\end{aligned} \quad (12)$$

Where h - the integration step, i - the number of integration steps.

To standardise the format of the results for both stable and unstable cases and for convenience, the following symbols are used:

$$\begin{cases}\hat{A}(\bar{t},\tau) = \begin{cases}A\big(\bar{x}(\tau - T - \bar{t}_{unst})\big)\\ -A(\bar{x}(\bar{t}_{st} - \tau + T))\end{cases}\\ \hat{f}(\bar{t},\tau) = \begin{cases}f(\bar{x}(\tau - T - \bar{t}_{unst}),\tau)\\ -f(\bar{x}(\bar{t}_{st} - \tau + T),\tau)\end{cases}\end{cases} \quad (13)$$

Similarly, we will derive a formula for calculating the matricant using the rule of rectangles, while taking property (9) into account:

$$\begin{aligned}\hat{\Omega}_0^{h\cdot i} = \Bigg(E + \frac{h}{2}\Big(\hat{A}\left(\bar{t},h\cdot(i-1)\right) + \hat{A}\left(\bar{t},h\cdot i\right)\Big) + \\ + \frac{h}{4}\Big(\hat{A}\left(\bar{t},h\cdot(i-1)\right) + \hat{A}\left(\bar{t},h\cdot i\right)\Big)^2\Bigg)\cdot\hat{\Omega}_0^{h\cdot(i-1)}\end{aligned} \quad (14)$$

Using expressions (10) - (14) and heteroclinic solutions, we can construct split manifolds. In addition to the numerical-analytical method, we have implemented the construction of phase trajectory evolution based on the direct integration of a system of differential equations.

The main indicator of chaotic dynamics emerging is the intersection of split manifolds. To detect intersection points, we use a vector function.

$$d(\bar{t}_{unst},\bar{t}_{st}) = x_{st}^{i}(\bar{t}_{st}^{i}) - x_{unst}^{j}(\bar{t}_{unst}^{j}) \quad (15)$$

IV. NUMERICAL SIMULATION

Fig. 2 shows the split manifolds of one heteroclinic branch. The trajectories obtained by the maritcant method and the phase trajectory evolution method practically coincide. Fig. 3 shows the manifolds for four heteroclinic trajectories, clearly showing the intersections of the perturbed manifolds and the splitting of the heteroclinic points. The intersections of the perturbed manifolds clearly visible.

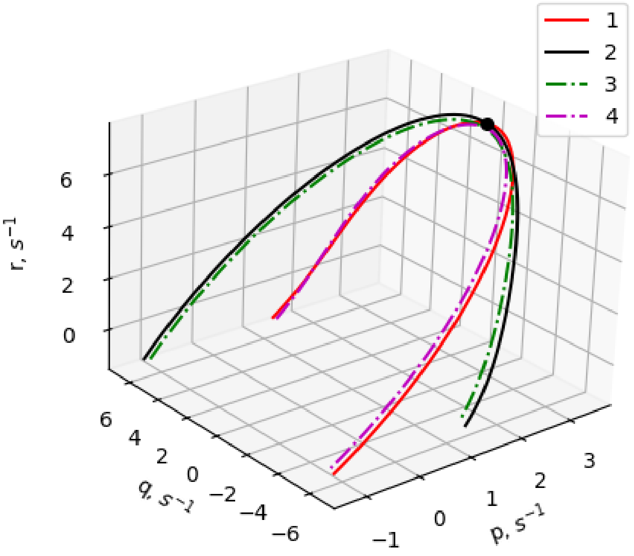

Fig. 2. The splitted heteroclinic trajectory, 1 – unstable manifold calculated using the matricant method, 2 – stable manifold calculated using the matricant method, 3 – unstable manifold obtained by direct integration of the system, 4 – stable manifold obtained by direct integration of the system

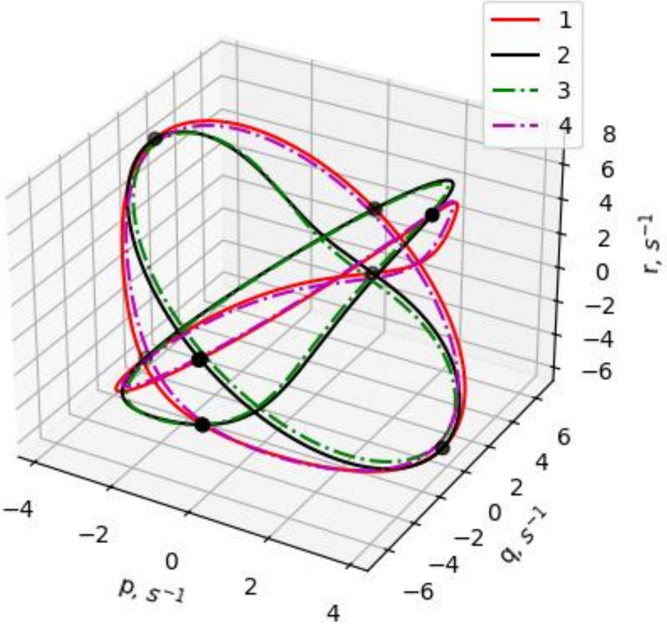

Fig. 3. Split heteroclinic trajectories, 1 – unstable manifold calculated using the matricant method, 2 – stable manifold calculated using the matricant method, 3 – unstable manifold obtained by direct integration of the system, 4 – stable manifold obtained by direct integration of the system

TABLE I. SYSTEM PARAMETERS

Parameters	**Unit of Measurement**	**Value**
A_1	$kg \cdot m^2$	5
A_2	$kg \cdot m^2$	15
B_2	$kg \cdot m^2$	8
C_1	$kg \cdot m^2$	4
C_2	$kg \cdot m^2$	6
μ	$N \cdot m$	96
ε		0.04

TABLE II. INITIAL CONDITIONS

Parameters	**Unit of Measurement**	**Value**
p_0	s^{-1}	3.5
q_0	s^{-1}	0
r_0	s^{-1}	6.68
Δ_0	$\frac{kg \cdot m^2}{s}$	2
ω_Δ	s^{-1}	20

V. THE DISCUSSION OF THE RESULTS

In this work, the matricant method is used to analyze the splitting of heteroclinic phase trajectories in the presence of a small internal periodic perturbation. An algorithm is constructed to calculate heteroclinic splitting. The split manifolds for all heteroclinic trajectories are determined numerically and their intersections are identified. These manifolds are formed as the initial images of the Poincaré section, obtained through direct numerical integration of the system.

Further refinement of the developed method is clearly necessary, as is an analysis of the limits of its applicability. Important aspects of the study include approaches to validation and verification of the method and its resulting results. It is also worth noting that, for such perturbed systems, analytical solutions in principle do not exist, as was demonstrated by H. Poincaré and later formulated by V.V. Kozlov in terms of the "non-integrability" of perturbed Hamiltonian systems [8-10].

At the present stage, the development of validation and verification schemes is only possible through numerical integration of perturbed equations with improved accuracy, which is a task for future research. Such a study of perturbed equations is no longer intended as a brief presentation of ideas at a conference, but rather as a self-contained scientific article. This is the plan for future work by the authors.

VI. CONCLUSION

The paper considers the dynamics of a gyrostat-satellite under the action of internal periodic perturbations. The matricant method is used to analyze chaotic dynamics. As can be seen from the calculations (Fig. 2-3), the stable and unstable manifolds of all heteroclinic trajectories intersect, which indicates the chaotic nature of the spacecraft's motion. The results obtained using the matricant method and the phase trajectory evolution method with numerical integration of the system's differential equations coincided with sufficient accuracy, with the deviation of the linearized system from the non-linearized system being less than 4%.

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